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THE PRESSURE OF A PUNCH IN THE FORM OF AN ELLIPTIC PARABOLOID ON AN ELASTIC LAYER OF FINITE THICKNESS†

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The problem of the indentation (without friction) of an absolutely solid body into an elastic layer is investigated. It is assumed that the diameter of the contact area which is unknown in advance, is small compared with the layer thickness. A model unilateral contact problem of the pressure on the elastic half-space of a punch with a surface which is close to an elliptic paraboloid is derived using the method of matched asymptotic expansions. The asymptotic solution of the model problem and the asymptotic form of the boundary of the contact area are constructed using Nazarov's method. A uniformly valid asymptotic representation is found for the density of the contact pressures. The asymptotic solution of the axisymmetric problem is written out in explicit form. © 2001 Elsevier Science Ltd. All rights reserved.

1. FORMULATION OF THE PROBLEM

We will assume that a punch in the form of an elliptic paraboloid

$$x_3 = -\Phi(x_1, x_2); \quad \Phi(x_1, x_2) = (2R_1)^{-1} x_1^2 + (2R_2)^{-1} x_2^2 \tag{1.1}$$

is impressed into an elastic layer (with shear modulus μ and Poisson's ratio ν) of thickness H, fixed to a rigid base ($x_3 = H$), to a depth δ_0 without friction. Here, R_1, R_2 are the radii of curvature of the principal normal cross-sections of the face of the stamp at its vertex ($R_1 \geqslant R_2$). We assume that the quantities δ_0 and R_1, R_2 are small compared with H. A small positive parameter is denoted by ε and we put

$$R_1 = \varepsilon R_1^*, \quad R_2 = \varepsilon R_2^*, \quad \delta_0 = \varepsilon \delta_0^* \tag{1.2}$$

where the magnitudes of δ_0^* and R_1^*, R_2^* are comparable with H.

The vector $\mathbf{u} = (u_1, u_2, u_3)$ of the displacements of the points of the elastic layer satisfy the unilateral contact problem (see [1, 2], etc.)

$$L(\nabla_x)\mathbf{u}(\varepsilon;\mathbf{x}) \equiv -\mu\nabla_x \cdot \nabla_x \mathbf{u}(\varepsilon;\mathbf{x}) - \frac{\mu}{1-2\nu}\nabla_x \nabla_x \cdot \mathbf{u}(\varepsilon;\mathbf{x}) = 0, \quad x_3 \in (0, H) \tag{1.3}$$

$$\sigma_{31}(\mathbf{u};\mathbf{x}) = \sigma_{32}(\mathbf{u};\mathbf{x}) = 0, \quad x_3 = 0 \tag{1.4}$$

$$\begin{gathered} u_3(\varepsilon;\mathbf{x}) \geqslant \varepsilon\delta_0^* - \Phi_\varepsilon(x_1, x_2), \quad \sigma_{33}(\mathbf{u};\mathbf{x}) \leqslant 0 \\ \left[u_3(\varepsilon;\mathbf{x}) - \varepsilon\delta_0^* + \Phi_\varepsilon(x_1, x_2)\right]\sigma_{33}(\mathbf{u};\mathbf{x}) = 0, \quad x_3 = 0 \end{gathered} \tag{1.5}$$

$$\mathbf{u}(\varepsilon;\mathbf{x}) = 0, \quad x_3 = H \tag{1.6}$$

Here $L(\nabla_x)$ is the Lamé operator and $\sigma_{3j}(\mathbf{u})$ are the components of the stress tensor. The contact area (where the equality sign holds in the first inequality of (1.5)) is not known in advance and is determined by the condition that the contact pressures

$$p(x_1, x_2) = -\sigma_{33}(\mathbf{u}; x_1, x_2, 0) \tag{1.7}$$

are positive.

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The axisymmetric problem ($R_1 = R_2$) has been studied in detail in [3]. The asymptotic solution of problem (1.3)–(1.6) is given by the "large λ" method [4] in [5] by assuming that the contact area is bounded by an ellipse. This approach was also used when calculating the pressure of a punch in the form of an elliptic paraboloid on the boundary of an elastic three-dimensional wedge [6].

Another solution of the problem in question was constructed in [7] using Aleksandrov's method [8]. An approximate equation was derived in [7], by approximating the regular component of the kernel of the integral operator of the contact problem by a second-degree polynomial, for determining the density of the contact pressures $p^0(x_1, x_2)$, which enables the following exact solution (which vanishes on the contour of the elliptic contact area) to be constructed

$$p^0(x_1, x_2) = \frac{3Q}{2\pi a^2\sqrt{1-e^2}}\sqrt{1-\frac{x_1^2}{a^2}-\frac{x_2^2}{a^2(1-e^2)}} \tag{1.8}$$

To determine the major semiaxis a, the eccentricity e and the magnitude Q of the force acting on the punch, the following system of equations was obtained [7]

$$\delta_0 = \frac{3\tilde{Q}}{2a}\left(\mathbf{K}(e) - \frac{2a_0 a}{3H} - \frac{2a_1 a^3}{15H^3}(2-e^2)\right), \quad \tilde{Q} = \frac{(1-\nu)}{2\pi\mu}Q \tag{1.9}$$

$$\frac{1}{R_1} - 2a_1\frac{\tilde{Q}}{H^3} = \frac{3\tilde{Q}}{a^3}\mathbf{D}(e), \quad \frac{1}{R_2} - 2a_1\frac{\tilde{Q}}{H^3} = \frac{3\tilde{Q}}{a^3}\frac{\mathbf{B}(e)}{1-e^2} \tag{1.10}$$

Here

$$\mathbf{D}(e) = e^{-2}[\mathbf{K}(e) - \mathbf{E}(e)], \quad \mathbf{B}(e) = e^{-2}\left[\mathbf{E}(e) - (1-e^2)\mathbf{K}(e)\right]$$

where **K** and **E** are complete elliptic integrals of the first and second kind. The integral representation

$$a_m = \frac{(-1)^m}{2^{2m}(m!)^2}\int_0^\infty [1 - L(u)]u^{2m}du \tag{1.11}$$

is known for the coefficients a_0 and a_1 (see [3, 9]).

In the case of a layer which is rigidly bonded to a non-deformable base,

$$L(u) = \frac{2\varkappa \operatorname{sh} 2u - 4u}{2\varkappa \operatorname{ch} 2u + 1 + \varkappa^2 + 4u^2}, \quad \varkappa = 3 - 4\nu$$

The numerical values of the coefficients a_m for various values of Poisson's ratio ν are available in Table 1.2 in [9]. Independently of the earlier results in [7], a solution, which is similar to (1.8)–(1.10), of the unilateral contact problem of the indentation of a punch (1.1) into the plane section of an elastic body was found in [10] by the method of the matched asymptotic expansions [11–13]. In particular, it was found that formulae (1.8)–(1.10), with appropriate values of the coefficients a_0 and a_1, give an approximate solution of the problem of the pressure of a punch in the form of an elliptic paraboloid on the centre of an elastic hemispere or a circular plate subject to the condition of axial symmetry of their fixing. The asymptotic form of the solution of the resulting problem (1.9), (1.0) was given in [10] under assumption (1.2).

It is also clear that solution (1.9), (1.10) also holds in the case of an elastic layer of finite thickness, coupled to an elastic half-space. The solution of the axisymmetric problem (for the values of the coefficients a_m, see Table 2 [14]) has been constructed in [14] by the "large λ" method. Formulae are obtained in [15] which are convenient for practical calculations. A numerical solution of the non-axisymmetric problem for punch (1.1) is given in [16].

It has been shown using the example of a linear contact problem [17] that solution (1.8), of the so-called "combined" [18] integral equation of the contact problem, which is obtained by a method previously described in [8, 7], possesses greater accuracy than the expansion in [5] found using the "large λ" method . At the same time, an attempt to refine the asymptotic form using the method described in [7] or [10] leads to the need to solve a unilateral contact problem for a punch bounded by a fourth-order surface. Generally speaking, the exact solution of this problem is unknown (see [19, 20, 5]).

In this paper (which is a continuation of [10]), an asymptotic method [21], which was previously used to solve the contact problem in [22], is employed to construct the solution of the above-mentioned refined model of the unilateral contact problem (formulated in Section 3). It is possible, by this route, to describe successfully the variation of the elliptic contact area and to obtain a uniformly valid asymptotic representation for the density of the contact pressures in explicit form.

2. THE DISPLACEMENT FIELD FAR FROM THE CONTACT AREA

We will denote the singular solutions of problems of the action in the boundary layer of a point force and the mth order polymoments which are applied at the origin of the coordinate system by $\mathbf{G}(\mathbf{x})$ and $\mathbf{G}^{(m,n)}(\mathbf{x})$. The expansion

$$\mathbf{G}(\mathbf{x})=\mathbf{T}(\mathbf{x})+\mathbf{g}(\mathbf{x}),\quad \mathbf{G}^{(m,n)}(\mathbf{x})=\mathbf{S}^{(m,n)}(\mathbf{x})+\mathbf{g}^{(m,n)}(\mathbf{x}) \tag{2.1}$$

in the singular and regular components holds. Here $\mathbf{T}(\mathbf{x})$ is the solution of the Boussinesq problems of the loading of an elastic half-space $x_3 \geqslant 0$ with a single point force directed along the Ox_3 axis

$$\mathbf{S}^{(m,n)}(\mathbf{x})=\partial^m\mathbf{T}(\mathbf{x})/\partial x_1^{m-n}\partial x_2^n$$

Explicit expressions for the vector functions $\mathbf{g}(\mathbf{x})$ and $\mathbf{g}^{(m,n)}(\mathbf{x})$ can be obtained using a Fourier transformation (see [9], for example).

Following the approach described previously [10], we approximate the vector of the displacements of the points of an elastic semi-infinite body far from the point of contact by the sum

$$\boldsymbol{v}(\varepsilon;\mathbf{x})=\mathcal{Q}\mathbf{G}(\mathbf{x})+\sum_{n=0,2}\mathcal{M}_{2,n}\mathbf{G}^{(2,n)}(\mathbf{x})+\sum_{n=0,2,4}\mathcal{M}_{4,n}\mathbf{G}^{(4,n)}(\mathbf{x}) \tag{2.2}$$

With the aim of writing out the asymptotic form of $v(\varepsilon;\mathbf{x})$ when $|\mathbf{x}|=(x_1^2+x_2^2+x_3^2)^{1/2}\to 0$, we use the formula

$$\mathbf{g}(\mathbf{x})=\mathbf{g}(0)+\sum_{m=1}^{4}\sum_{k=1}^{N(m)}g_{m,k}\mathbf{V}_k^m(\mathbf{x})+O\left(|\mathbf{x}|^5\right) \tag{2.3}$$

where $\mathbf{V}_1^m(\mathbf{x}),\ldots,\mathbf{V}_{N(m)}^m(\mathbf{x})$ is a basis in the space of the homogeneous vector polynomials of degree m which satisfy the homogeneous Lamé system (1.3) in the half-space $x_3>0$ and the boundary condition that its boundary is stress-free: $N(m)=3(m+1)$. Explicit expressions for the vectors $\mathbf{V}_k^m(\mathbf{x})$ (for $m=1,2$, see [18]) are not subsequently needed. We also note that, by virtue of the axial symmetry, many of the coefficients $g_{m,k}$ in (2.3) are equal to zero.

The following formula is necessary in order to derive the resulting unilateral contact problem for the inner asymptotic expansion

$$\frac{2\pi\mu}{1-\nu}g_3(x_1,x_2,0)=A_0+A_{2,0}x_1^2+A_{0,2}x_2^2+A_{4,0}x_1^4+A_{2,2}x_1^2x_2^2+A_{0,4}x_2^4+\ldots \tag{2.4}$$

$$A_0=-\frac{a_0}{H},\quad A_{2,0}=A_{0,2}=-\frac{a_1}{H^3},\quad A_{4,0}=A_{0,4}=-\frac{a_2}{H^5},\quad A_{2,2}-\frac{2a_2}{H^5} \tag{2.5}$$

where a_0, a_1 and a_2 are the coefficients of (1.1)

Expansions similar to (2.3) and (2.4) hold in the case of the vector function $\mathbf{g}^{(m,n)}(\mathbf{x})$. The coefficients

$$A_0^{(2,0)}=A_0^{(2,2)}=-\frac{2a_1}{H^3},\quad A_{2,0}^{(2,0)}=A_{0,2}^{(2,2)}=-\frac{12a_2}{H^5},\quad A_{0,2}^{(2,0)}=A_{2,0}^{(2,2)}=-\frac{4a_2}{H^5} \tag{2.6}$$

$$A_0^{(4,0)}=A_0^{(4,4)}=-\frac{24a_2}{H^5},\quad A_0^{(4,2)}=-\frac{8a_2}{H^5} \tag{2.7}$$

will be required later.

According to assumption (1.2), when $\varepsilon\to 0$ we establish the following orders for the coefficients in the outer asymptotic expansion (2.2)

$$\mathfrak{Q} = \varepsilon^2 \mathfrak{Q}^*, \quad \mathcal{M}_{2,n} = \varepsilon^4 \mathcal{M}_{2,n}^*, \quad \mathcal{M}_{4,n} = \varepsilon^6 \mathcal{M}_{4,n}^* \tag{2.8}$$

Relations (2.8) follow from formulae in Hertz's theory (in particular, see [23, Chapter 5, Section 6.5]).

3. ASYMPTOTIC MODELLING OF THE PRESSURE OF A PUNCH IN THE FORM OF AN ELLIPTIC PARABOLOID ON AN ELASTIC LAYER

In the neighbourhood of the punch, we change to the "extended" coordinates

$$\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3); \quad \xi_i = \varepsilon^{-1} x_i \tag{3.1}$$

and formulate the problem for the inner asymptotic expansion $\mathbf{w}(\varepsilon; \xi)$ in the half-space $\xi_3 \geqslant 0$. Relations (1.3) and (1.4) give

$$L(\nabla_\xi)\mathbf{w}(\varepsilon;\boldsymbol{\xi}) = 0, \quad \xi_3 < 0; \quad \sigma_{31}(\mathbf{w};\boldsymbol{\xi}) = \sigma_{32}(\mathbf{w};\boldsymbol{\xi}) = 0, \quad \xi_3 = 0 \tag{3.2}$$

We will restrict the unilateral contact boundary condition (1.5) to the domain

$$\omega^* = \left\{(\xi_1, \xi_2): \ \left(2\delta_0^* R_1^*\right)^{-1} \xi_1^2 + \left(2\delta_0^* R_2^*\right)^{-1} \xi_2^2 < 1\right\}$$

which necessarily includes the required contact area (outside the domain ω^* the surface of the punch is located above the level of the unperturbed boundary of the elastic layer). We have

$$\sigma_{33}(\mathbf{w}; \boldsymbol{\xi}', 0) = 0, \quad \boldsymbol{\xi}' = (\xi_1, \xi_2) \notin \omega^* \tag{3.3}$$

$$\begin{gathered} w_3(\varepsilon; \boldsymbol{\xi}', 0) \geqslant \varepsilon\left(\delta_0^* - \Phi^*(\xi_1, \xi_2)\right) \quad \sigma_{33}(\mathbf{w}; \boldsymbol{\xi}', 0) \leqslant 0, \\ \left[w_3(\varepsilon; \boldsymbol{\xi}', 0) - \varepsilon\left(\delta_0^* - \Phi^*(\xi_1, \xi_2)\right)\right]\sigma_{33}(\mathbf{w}; \boldsymbol{\xi}', 0) = 0, \quad \boldsymbol{\xi}' \in \omega^* \end{gathered} \tag{3.4}$$

$$\Phi^*(\xi_1, \xi_2) = \left(2R_1^*\right)^{-1} \xi_1^2 + \left(2R_2^*\right)^{-1} \xi_2^2$$

Relations (3.2)–(3.4) are closed by the asymptotic condition for the behaviour of the vector $\mathbf{w}(\varepsilon; \xi)$ when $|\xi| \to \infty$, which we obtain by matching with outer expansion (2.2).

We substitute expression (2.8) into (2.2), take account of the asymptotic formulae of the type of (2.3) and introduce the extended coordinates (3.1). In the matching region $\{\mathbf{x}: \sqrt{\varepsilon}H/2 \leqslant |\mathbf{x}| \leqslant \sqrt{\varepsilon}H\}$, we therefore find

$$\begin{aligned} \mathbf{v}(\varepsilon; \varepsilon\boldsymbol{\xi}) = \varepsilon&\left[\mathfrak{Q}^* \mathbf{T}(\boldsymbol{\xi}) + \sum_{n=0,2} \mathcal{M}_{2,n}^* \mathbf{S}^{(2,n)}(\boldsymbol{\xi}) + \sum_{n=0,2,4} \mathcal{M}_{4,n}^* \mathbf{S}^{(4,n)}(\boldsymbol{\xi})\right] + \\ &+ \varepsilon^2\left[\mathbf{V}^*(\varepsilon; \boldsymbol{\xi}) + O\left(\varepsilon^5 |\boldsymbol{\xi}|^5\right)\right] \end{aligned} \tag{3.5}$$

Here, $\mathbf{V}^*(\varepsilon; \xi)$ is a fourth-degree vector polynomial

$$\begin{aligned} \mathbf{V}^*(\varepsilon; \boldsymbol{\xi}) = \mathfrak{Q}^* &\left[\mathbf{g}(0) + \sum_{m=1}^{4} \varepsilon^m \sum_{k=1}^{N(m)} g_{m,k} \mathbf{V}_k^m(\boldsymbol{\xi})\right] + \\ &+ \varepsilon^2 \sum_{n=0,2} \mathcal{M}_{2,n}^* \left[\mathbf{g}^{(2,n)}(0) + \sum_{m=1}^{2} \varepsilon^m \sum_{k=1}^{N(m)} g_{m,k}^{(2,n)} \mathbf{V}_k^m(\boldsymbol{\xi})\right] + \varepsilon^4 \sum_{n=0,2,4} \mathcal{M}_{4,n}^* \mathbf{g}^{(4,n)}(0) \end{aligned} \tag{3.6}$$

Taking account of relation (3.5), we derive the above-mentioned condition

$$\mathbf{w}(\varepsilon;\boldsymbol{\xi}) = \varepsilon^2\mathbf{V}^*(\varepsilon;\boldsymbol{\xi}) + \varepsilon\left[\mathfrak{Q}^*\mathbf{T}(\boldsymbol{\xi}) + \sum_{n=0,2}\mathcal{M}^*_{2,n}\mathbf{S}^{(2,n)}(\boldsymbol{\xi}) + \sum_{n=0,2,4}\mathcal{M}^*_{4,n}\mathbf{S}^{(4,n)}(\boldsymbol{\xi})\right] + O\left(|\boldsymbol{\xi}|^{-7}\right),\quad |\boldsymbol{\xi}|\to\infty \tag{3.7}$$

The solution of problem (3.2)–(3.4), (3.7) can be written in the form

$$\mathbf{w}(\varepsilon;\boldsymbol{\xi}) = \varepsilon^2\mathbf{V}^*(\varepsilon;\boldsymbol{\xi}) + \varepsilon\mathbf{W}(\varepsilon;\boldsymbol{\xi}) \tag{3.8}$$

where $\mathbf{W}(\varepsilon;\xi)$ is a vector function which vanishes when $|\xi|\to\infty$ and satisfies relations (3.2) and (3.3) and the boundary condition

$$\begin{gathered} W_3(\varepsilon;\boldsymbol{\xi}',0) \geqslant \delta_0^* - \Phi^*(\xi_1,\xi_2) - \varepsilon V_3^*(\varepsilon;\boldsymbol{\xi}),\quad \sigma_{33}(\mathbf{W};\boldsymbol{\xi}',0)\leqslant 0 \\ \left[W_3(\varepsilon;\boldsymbol{\xi}',0) - \delta_0^* + \Phi^*(\xi_1,\xi_2) + \varepsilon V_3^*(\varepsilon;\boldsymbol{\xi}',0)\right]\sigma_{33}(\mathbf{W};\boldsymbol{\xi}',0) = 0,\quad \boldsymbol{\xi}'\in\omega^* \end{gathered} \tag{3.9}$$

As a consequence of matching condition (3.7), the equations for determining the quantities $\mathfrak{Q}^*$, $\mathcal{M}^*_{2,n}$ $(n = 0, 2)$ and $\mathcal{M}^*_{4,n}$ $(n = 0, 2, 4)$ will be

$$\mathfrak{Q}^* = -\iint_{\omega_\varepsilon^*}\sigma_{33}(\mathbf{W};\xi_1,\xi_2,0)d\xi_1 d\xi_2 \tag{3.10}$$

$$\mathcal{M}^*_{m,n} = -\frac{(-1)^m}{m!}C_m^n\iint_{\omega_\varepsilon^*}\xi_1^{m-n}\xi_2^n\sigma_{33}(\mathbf{W};\xi_1,\xi_2,0)d\xi_1 d\xi_2 \tag{3.11}$$

Here ω_ε^* is the contact area corresponding to the displacement vector $\mathbf{w}(\varepsilon;\xi)$ and C_m^n is a binomial coefficient.

Thus, in the case of the vector function $\mathbf{W}(\varepsilon;\xi)$, the unilateral contact problem of the pressure on an elastic half-space of a punch with a face represented by a fourth-degree polynomial is obtained.

Let us put

$$\mathfrak{Q}^* = Q^* + \varepsilon^5 q^*,\quad \mathcal{M}^*_{m,n} = M^*_{m,n} + \varepsilon^5 m^*_{m,n},\quad m = 2,4 \tag{3.12}$$

Now, while remaining within the accuracy to which problem (3.2)–(3.4), (3.7) is valid, we will simplify the right-hand side of the first inequality in (3.9), rewriting it as follows:

$$\delta_0^* - \Phi^*(\xi_1,\xi_2) - \varepsilon V_3^*(\varepsilon;\xi_1,\xi_2,0) \simeq \delta_\varepsilon^* - \Phi_\varepsilon^*(\xi_1,\xi_2) - \varepsilon^5\varphi_0^*(\xi_1,\xi_2) \tag{3.13}$$

From relations (3.6) and (2.3), (2.4), taking (3.12) into account, we obtain

$$\delta_\varepsilon^* = \delta_0^* - \varepsilon\tilde{Q}^* A_0 - \varepsilon^3\sum_{n=0,2}\tilde{M}^*_{2,n}A_0^{(2,n)} \tag{3.14}$$

$$\Phi_\varepsilon^*(\xi_1,\xi_2) = \left[\left(2R_1^*\right)^{-1} + \varepsilon^3\tilde{Q}^* A_{2,0}\right]\xi_1^2 + \left[\left(2R_2^*\right)^{-1} + \varepsilon^3\tilde{Q}^* A_{0,2}\right]\xi_2^2 \tag{3.15}$$

A tilde over the symbols Q^* and $M^*_{2,n}$ denotes their multiplication by $(1-\nu)(2\pi\mu)^{-1}$ (see the second formula of (1.9)).

Since only terms of the order of ε^5 compared with unity have been retained when setting up the left-hand side of relation (3.13), we put

$$\begin{aligned} \varphi_0^*(\xi_1,\xi_2) &= \tilde{Q}_0^*\left(A_{4,0}\xi_1^4 + A_{2,2}\xi_1^2\xi_2^2 + A_{0,4}\xi_2^4\right) + \\ &+ \sum_{n=0,2}\tilde{M}^{*0}_{2,n}\left(A_{2,0}^{(2,n)}\xi_1^2 + A_{0,2}^{(2,n)}\xi_2^2\right) + \sum_{n=0,2,4}\tilde{M}^{*0}_{4,n}A_0^{(4,n)} \end{aligned} \tag{3.16}$$

The quantities $\tilde{Q}_0^*$, $\tilde{M}^{*0}_{2,n}$ and $\tilde{M}^{*0}_{4,n}$ correspond to the zeroth approximation (we must put $\varepsilon = 0$ in (3.9)) and they are calculated using formulae (3.10) and (3.11) for the Hertzian density of the contact pressures (1.8). We have

$$\tilde{M}_{2,0}^{*0} = 10^{-1}\tilde{Q}_0^*\left(a_0^*\right)^2, \quad \tilde{M}_{2,2}^{*0} = \tilde{M}_{2,0}^{*0}\left(1-e_0^2\right)$$

$$\tilde{M}_{4,0}^{*0} = \frac{\tilde{Q}_0^*}{560}\left(a_0^*\right)^4\left(1+\sqrt{1-e_0^2}\right), \quad \tilde{M}_{4,2}^{*0} = \frac{\tilde{Q}_0^*}{140}\left(a_0^*\right)^4\left(1-e_0^2\right), \quad \tilde{M}_{4,4}^{*0} = \tilde{M}_{4,0}^{*0}\left(1-e_0^2\right)^{3/2} \tag{3.17}$$

where a_0^* and e_0 are, respectively, the major semiaxis and the eccentricity of the Hertzian contact area (when the layer is replaced by a half-space).

4. ASYMPTOTIC SOLUTION OF THE MODEL UNILATERAL CONTACT PROBLEM

We shall seek a solution $\mathbf{W}(\varepsilon;\ \xi)$, which vanishes at infinity, of problem (3.2), (3.3), (3.9), taking (3.13)–(3.16) into account, using the method described previously in [21] in the form of two asymptotic approximations: an outer expansion which holds far from the contour Γ_ε^* of the required contact area ω_ε^* and an inner expansion for the small neighbourhood Γ_ε^*.

We will denote the sum

$$\mathcal{V}(\varepsilon;\boldsymbol{\xi}) = \mathcal{V}^0(\varepsilon;\boldsymbol{\xi}) + \varepsilon^5\mathcal{V}^1(\varepsilon;\boldsymbol{\xi}) \tag{4.1}$$

as the outer asymptotic expansion for the vector $\mathbf{W}(\varepsilon;\ \xi)$, where $\mathcal{V}^0(\varepsilon;\ \xi)$ is the solution of the contact problem on the indentation into an elastic half-space $\xi_3 \geqslant 0$ to a depth δ_ε^* of a punch in the form of an elliptic paraboloid $\xi_3 = -\Phi_\varepsilon^*\ (\xi_1,\ \xi_2)$ (for the expressions for δ_ε^* and $\Phi_\varepsilon^*\ (\xi_1,\ \xi_2)$, see (3.14) and (3.15)). We will write the vector function $\mathcal{V}^0(\varepsilon;\ \xi)$ in the form of the generalized potential of a simple layer

$$\mathcal{V}^0(\varepsilon;\boldsymbol{\xi}) = \iint\limits_{\omega_0^*} p^0(\varepsilon;\eta_1,\eta_2)\mathbf{T}(\xi_1-\eta_1,\xi_2-\eta_2,\xi_3)d\eta_1 d\eta_2 \tag{4.2}$$

with a contact pressure distribution density

$$p^0(\varepsilon;\xi_1,\xi_2) = p_0 H^*(\xi_1,\xi_2) \tag{4.3}$$

$$p_0 = \frac{3Q^*}{2\pi\left(a^*\right)^2\sqrt{1-e^2}}, \quad H^*(\xi_1,\xi_2) = \sqrt{1-\frac{\xi_1^2}{\left(a^*\right)^2}-\frac{\xi_2^2}{\left(a^*\right)^2\left(1-e^2\right)}}$$

The elliptic contact area ω_0^* with contour Γ_0^* (the dependence of ω_0^* and Γ_0^* on the parameter ε is not indicated) corresponds to displacement vector (4.2). The major semiaxis and the eccentricity of the ellipse Γ_0^* are denoted by a^* and e. To determine Q^*, a^* and e using Hertz' formulae, we have the following system of equations

$$\delta_0^* - \varepsilon\tilde{Q}^* A_0 - \varepsilon^3 \sum_{n=0,2} \tilde{M}_{2,n}^* A_0^{(2,n)} = \frac{3\tilde{Q}^*}{2a^*}\mathbf{K}(e) \tag{4.4}$$

$$\frac{1}{R_1^*} + \varepsilon^3 2\tilde{Q}^* A_{2,0} = \frac{3\tilde{Q}^*}{\left(a^*\right)^3}\mathbf{D}(e), \quad \frac{1}{R_2^*} + \varepsilon^3 2\tilde{Q}_0^* A_{0,2} = \frac{3\tilde{Q}^*}{\left(a^*\right)^3}\frac{\mathbf{B}(e)}{1-e^2} \tag{4.5}$$

Finally, taking account of relations (3.12), (4.1) and (3.10), (3.11), we supplement Eqs (4.4) and (4.5) with the relations for the polymoments occurring in (4.4),

$$M_{2,0}^* = 10^{-1}Q^*\left(a_0^*\right)^2, \quad M_{2,2}^* = 10^{-1}Q^*\left(a_0^*\right)^2\left(1-e^2\right) \tag{4.6}$$

Substituting expressions (4.6) into Eq. (4.4), we reduce system (4.4), (4.5) to the form (1.10), (1.11).

We will now describe the behaviour of the integral (4.2) and its density (4.3) in the neighbourhood Γ_0^* of the domain ω_0^* (see [24, 25], etc.). We define the contour Γ_0^* by the equations

$$\xi_1 = a^* \cos\sigma, \quad \xi_2 = a^* \sqrt{1-e^2} \sin\sigma$$

where σ is a parameter. Then, the unit vector of the inward normal (with respect to the domain ω_0^*) to Γ_0 is

$$\mathbf{n}(\sigma) = -\left(1 - e^2 \cos^2\sigma\right)^{-1/2}\left[\sqrt{1-e^2} \cos\sigma \, \mathbf{e}_1 + \sin\sigma \, \mathbf{e}_2\right]$$

In the spatial neighbourhood of the contour Γ_0^*, we introduce the system of coordinates ν, ξ_3, σ, associated with the Cartesian coordinates by the formulae

$$\xi_1 = a^* \cos\sigma + \nu n_1^0(\sigma), \quad \xi_2 = a^* \sqrt{1-e^2} \sin\sigma + \nu n_2^0(\sigma)$$

Moreover, in the $\pi(\sigma)$ planes, which are orthogonal to Γ_0^*, we introduce the polar coordinates r and φ such that

$$\nu = r\cos\varphi, \quad \xi_3 = r\sin\varphi, \quad \varphi \in [0, \pi]$$

By simple calculations for the density of the contact pressures (4.3) we find

$$p^0(\varepsilon; \xi_1, \xi_2) = -(2\pi)^{-1/2} k^{0*}(\sigma) r^{1/2} + O\left(r^{3/2}\right), \quad r \to 0 \quad (\varphi = 0) \tag{4.7}$$

$$k^{0*}(\sigma) = -\frac{3Q^*}{\sqrt{\pi}\left(a^*\right)^{5/2}} \frac{\left(1 - e^2\cos^2\sigma\right)^{1/4}}{\left(1-e^2\right)^{3/4}} \tag{4.8}$$

By introducing the projections onto the unit vectors of the local system of coordinates

$$\mathcal{V}_n^0 = n_1^0 \mathcal{V}_1^0 + n_2^0 \mathcal{V}_2^0, \quad \mathcal{V}_t^0 = n_2^0 \mathcal{V}_1^0 - n_1^0 \mathcal{V}_2^0$$

in the plane $\pi(\sigma)$ for the vector (4.2) when $r \to 0$, we can derive the following expansion

$$\left(\mathcal{V}_n^0 \mathbf{n}^0(\sigma) + \mathcal{V}_3^0 \mathbf{e}_3\right)(\varepsilon; r, \varphi, \sigma) = \sum_{i=1}^{2} \sum_{k=0,2} B_{i,k}^0(\sigma) \mathbf{X}^{i,k}(r,\varphi) + k^{0*}(\sigma)\mathbf{X}^{1,3}(r,\varphi) + O\left(r^3\right) \tag{4.9}$$

Here, $\mathbf{X}^{1,0}$, $\mathbf{X}^{2,0}$ and $\mathbf{X}^{2,2}(r, \varphi)$ are the translational displacements along the unit vectors $\mathbf{n}^0$, $\mathbf{e}_3$ and the rotation with respect to the unit vector $\mathbf{t}^0$ (tangential to Γ_0^*). The quantity $2\mu(1-\nu)^{-1} B_{1,2}^0$ has the meaning of the intensity of the stretching of the elastic half-space at the points of the contour Γ_0^* in the direction of its normal. In (4.9), we have used the notation previously adopted in [25] in which

$$\mathbf{X}^{1,3}(r,\varphi) = r^{3/2} \mathbf{\Phi}^{1,3}(\varphi)$$

where $\Phi^{1,3}(\varphi)$ is a vector with the polar components ($\varkappa = 3 - 4\nu$)

$$\mathbf{\Phi}_r^{1,3}(\varphi) = (2\pi)^{-1/2}(4\mu)^{-1}\left[\left(\tfrac{2}{3}\varkappa - 1\right)\cos(\varphi/2) + \tfrac{1}{3}\cos(5\varphi/2)\right]$$

$$\mathbf{\Phi}_\varphi^{1,3}(\varphi) = (2\pi)^{-1/2}(4\mu)^{-1}\left[\left(\tfrac{2}{3}\varkappa + 1\right)\sin(\varphi/2) - \tfrac{1}{3}\sin(5\varphi/2)\right]$$

The vector $\mathcal{V}^0(\varepsilon; \xi)$, which is the principal part of the asymptotic form (4.1), also determines the main approximation ω_0^* to the required contact area ω_ε^*. When constructing the vector function $\mathcal{V}^1(\varepsilon; \xi)$, we shall assume that the quantities Q^*, a^* and e are known. We emphasize (see [10]) that it is easy to obtain the asymptotic solution of (4.4)–(4.6) (apart from terms of the order of ε^5 inclusive). These formulae are not written here because of their length.

5. PERTURBATION OF THE HERTZ DISPLACEMENT FIELD FAR FROM THE CONTOUR OF THE CONTACT

Following [21], we denote the integral (4.2) with a density which satisfies the equation

$$\frac{1-\nu}{2\pi\mu}\iint_{\omega_0^*}\frac{p^1(\varepsilon;\eta_1,\eta_2)d\eta_1 d\eta_2}{\sqrt{(\xi_1-\eta_1)^2+(\xi_2-\eta_2)^2}}=-\varphi_0^*(\xi_1,\xi_2) \tag{5.1}$$

as the correction $\mathcal{V}^1(\varepsilon;\xi)$ in the outer asymptotic expansion (4.1) of the vector $\mathbf{W}(\varepsilon;\xi)$. The right-hand side of Eq. (5.1) is the fourth-degree polynomial (3.16) and, according to Galin's theorem (see [26, §9, Chapter 2]), the formula

$$p^1(x_1,x_2)=-\frac{2\pi\mu}{1-\nu}\frac{c_{00}^*+c_{20}^*\xi_1^2+c_{02}^*\xi_2^2+c_{40}^*\xi_1^4+c_{22}^*\xi_1^2\xi_2^2+c_{04}^*\xi_2^4}{H^*(\xi_1,\xi_2)} \tag{5.2}$$

holds for the solution of Eq. (5.1). The coefficients in the numerator of the fraction (5.2) can be expressed in explicit form in terms of the coefficients of the polynomial $\varphi_0^*(\xi_1,\xi_2)$ using formulae in [3, 9, 20]. Here, the quantity c_{00}^* is dimensionless and the dimensions of c_{20}^*, c_{02}^* and c_{40}^*, c_{22}^*, c_{04}^* are equal to L^{-2} and L^{-4} respectively, where L has the dimension of length.

The density (5.2), unlike (4.3), possesses a root singularity on approaching the contour Γ_0^*, that is,

$$p^1(\varepsilon;\xi_1,\xi_2)=-(2\pi)^{-1/2}K^{1*}(\sigma)r^{-1/2}+O(r^{1/2}),\quad r\to 0 \tag{5.3}$$

The quantity $K^{1*}(\sigma)$ has the meaning of the compressive stress intensity factor and is equal to

$$K^{1*}(\sigma)=\frac{2\pi\mu}{1-\nu}\frac{\sqrt{\pi}(1-e^2)^{1/4}(a^*)^{1/2}}{(1-e^2\cos^2\sigma)^{1/4}}\{c_{00}^*+(a^*)^2[c_{20}^*\cos^2\sigma+c_{02}^*(1-e^2)\sin^2\sigma]+$$

$$+(a^*)^4[c_{40}^*\cos^4\sigma+c_{22}^*(1-e^2)\cos^2\sigma\sin^2\sigma+c_{04}^*(1-e^2)^2\sin^4\sigma]\} \tag{5.4}$$

In turn, the formula

$$(\mathcal{V}_n^1\mathbf{n}^0(\sigma)+\mathcal{V}_3^1\mathbf{e}_3)(\varepsilon;r,\varphi,\sigma)=\sum_{i=1}^{2}B_{i,0}^1(\sigma)\mathbf{X}^{i,0}+K^{1*}(\sigma)\mathbf{X}^{1,1}(r,\varphi)+O(r) \tag{5.5}$$

holds for the component of the vector $\mathcal{V}^1(\varepsilon;\xi)$ in the plane $\pi(\sigma)$ when $r\to 0$. Here, $\mathbf{X}^{1,1}(r,\varphi)=r^{1/2}\Phi^{1,1}(\varphi)$, where $\Phi^{1,1}(\varphi)$ is a vector with polar components

$$\Phi_r^{1,1}(\varphi)=(2\pi)^{-1/2}(4\mu)^{-1}[(2\varkappa-1)\cos(\varphi/2)-\cos(3\varphi/2)]$$

$$\Phi_\varphi^{1,1}(\varphi)=(2\pi)^{-1/2}(4\mu)^{-1}[-(2\varkappa+1)\sin(\varphi/2)+\sin(3\varphi/2)]$$

Since, at infinity, the vector $\mathcal{V}(\varepsilon;\xi)$ must serve as an approximation to the vector $\mathbf{W}(\varepsilon;\xi)$, according to (3.7) and (3.8), when $|\xi|\to\infty$ we have

$$\mathcal{V}(\varepsilon;\xi)=\mathcal{Q}^*\mathbf{T}(\xi)+\sum_{n=0,2}\mathcal{M}_{2,n}^*\mathbf{S}^{(2,n)}(\xi)+\sum_{n=0,2,4}\mathcal{M}_{4,n}^*\mathbf{S}^{(4,n)}(\xi)+O(|\xi|^{-7}) \tag{5.6}$$

Now, substituting expressions (4.1) and (3.12) into relation (5.6), we find (see [18], for example) that the quantities q^* and $m_{m,n}^*$ from (3.12) are the integral characteristics (the resultant and polymoments of order m) of the density $p^1(\varepsilon;\xi_1,\xi_2)$.

By construction, the vector function $\mathcal{V}(\varepsilon;\xi)$ satisfies relations (3.2) and (3.3) and leaves a small discrepancy $o(\varepsilon^5)$ in the boundary condition of unilateral contact (3.9) within the domain ω_0^* far from its boundary Γ_0^*. In the neighbourhood of Γ_0^*, where the density $p^1(\varepsilon;\xi_1,\xi_2)$ is unbounded, the phenomenon of a plane boundary layer arises.

6. A PLANE BOUNDARY LAYER. VARIATION OF THE BOUNDARY OF THE ELLIPTIC CONTACT AREA

Let us assume that the contour Γ_ε^* of the required contact are a ω_ε^* in the local coordinates is described by the equation

$$\nu = h_\varepsilon^*(\sigma); \quad h_\varepsilon^*(\sigma) = \varepsilon^5 h^*(\sigma) \tag{6.1}$$

where $h^*(\sigma)$ is a function to be determined. In the planes $\pi(\sigma)$, which are orthogonal to Γ_0^*, we introduce the extended variables

$$\boldsymbol{\eta} = (\eta_1, \eta_2); \quad \eta_1 = \varepsilon^{-5}\nu, \quad \eta_2 = \varepsilon^{-5}\xi_3 \tag{6.2}$$

Then, the equation of the required boundary of the contact area takes the form $\eta_1 = h^*(\sigma)$.

Following the approach described previously in [21], we will seek the projection $\mathrm{Pr}_{\pi(\sigma)}\mathbf{W}(\varepsilon; \xi)$ of the vector $\mathbf{W}(\varepsilon; \xi)$ onto the plane $\pi(\sigma)$ in the neighbourhood of Γ_0^* in the form of the sum

$$\mathcal{W}(\varepsilon; \boldsymbol{\xi}) = \mathcal{W}^0(\sigma; \boldsymbol{\eta}) + \varepsilon^{5/2}\mathcal{W}^1(\sigma; \boldsymbol{\eta}) + \varepsilon^5\mathcal{W}^2(\sigma; \boldsymbol{\eta}) + \varepsilon^{15/2}\mathcal{W}^3(\sigma; \boldsymbol{\eta}) \tag{6.3}$$

Suppose $\rho = \varepsilon^{-5}r$ and φ are polar coordinates, corresponding to the Cartesian coordinates of (6.2). We now introduce another polar system of coordinates ρ_h and $\varphi_h \in [0, \pi]$ with the pole at the point $\eta_1 = h^*(\sigma)$, $\eta_2 = 0$. Then, the lowest term of expansion (6.3) turns out to be [21]

$$\mathcal{W}^3(\sigma; \boldsymbol{\eta}) = N(\sigma)\mathbf{X}^{1,3}(\rho_h, \varphi_h) \tag{6.4}$$

Note that vector function (6.4) is chosen such that the contact pressure, corresponding to (6.3), vanishes on the boundary of the contact area. Explicit expressions for $N(\sigma)$ and $h^*(\sigma)$ are determined by matching the outer and inner asymptotic expansions (4.1) and (6.3) respectively.

We now collect (4.9) and (5.5) into a single formula and, in the resulting expression, we change to the extended variables (6.2). As a result, we obtain the asymptotic formula

$$\begin{aligned}\mathrm{Pr}_{\pi(\sigma)}\mathcal{V}(\varepsilon; \boldsymbol{\xi}) &= \sum_{i=1}^{2} B_{i,0}^0(\sigma)\mathbf{X}^{i,0} + \varepsilon^5\sum_{i=1}^{2}(B_{i,2}^0(\sigma)\mathbf{X}^{i,2}(\rho,\varphi) + B_{i,0}^1(\sigma)\mathbf{X}^{i,0}) + \\ &+\varepsilon^{15/2}[k^{0*}(\sigma)\mathbf{X}^{1,3}(\rho,\varphi) + K^{1*}(\sigma)\mathbf{X}^{1,1}(\rho,\varphi)] + O(\varepsilon^{10}\rho, \varepsilon^{10}\rho^2)\end{aligned} \tag{6.5}$$

for the projection of the vector $\mathcal{V}(\varepsilon; \xi)$ onto the plane $\pi(\sigma)$.

Comparing (6.5) with (6.3), we find

$$\mathcal{W}^0(\sigma; \boldsymbol{\eta}) \equiv \sum_{i=1}^{2} B_{i,0}^0(\sigma)\mathbf{X}^{i,0}, \quad \mathcal{W}^1(\sigma; \boldsymbol{\eta}) \equiv 0$$

$$\mathcal{W}^2(\sigma; \boldsymbol{\eta}) \equiv \sum_{i=1}^{2} \left(B_{i,0}^1(\sigma)\mathbf{X}^{i,0} + B_{i,2}^0(\sigma)\mathbf{X}^{i,2}(\rho,\varphi)\right)$$

Next, by invoking the formulae

$$\partial\rho_h / \partial h^* = -\cos\varphi, \quad \partial\varphi_h / \partial h^* = \rho^{-1}\sin\varphi$$

which hold when $h^* = 0$ (see [27]) for large values of ρ_h, we derive

$$\rho_h^{3/2}\mathbf{\Phi}^{1,3}(\varphi_h) = \rho^{3/2}\mathbf{\Phi}^{1,3}(\varphi) - \frac{1}{2}h^*(\sigma)\rho^{1/2}\mathbf{\Phi}^{1,1}(\varphi) + O(\rho^{-1/2})$$

Using this relation, we match expansions (6.5) and (6.3) in the zone $\rho/a^* = O(\varepsilon^{-1/3})$ where the error in formula (6.5) proves to be small compared with the terms in (6.3). We therefore arrive at the equalities

$$N(\sigma) = k^{0*}(\sigma), \quad -\tfrac{1}{2}h^*(\sigma)N(\sigma) = K^{1*}(\sigma)$$

from which the relation

$$h^*(\sigma) = -2K^{1*}(\sigma)/k^{0*}(\sigma) \tag{6.6}$$

follows.

Formulae (6.6) and (6.1) largely determine the position of the boundary Γ_ε^*. Note that, in much the same way as the higher terms of expansion (6.3) did not participate when deriving equality (6.6), in the case of the functions $\mathcal{V}_t^0(\varepsilon; r, \varphi, \sigma) + \varepsilon^5\mathcal{V}_t^1(\varepsilon; r, \varphi, \sigma)$, the boundary layer function of the form (6.3) with zero second and fourth terms is independently finely adjusted.

7. THE ASYMPTOTIC FORM OF THE CONTACT PRESSURE

According to representation (4.1), the pressure under the punch far from Γ_ε^* is calculated using the formula

$$p(\varepsilon; \xi_1, \xi_2) \simeq p^0(\varepsilon; \xi_1, \xi_2) + \varepsilon^5 p^1(\varepsilon; \xi_1, \xi_2) \tag{7.1}$$

The terms are written out in (4.3) and (5.2).

On the basis of equalities (6.3) and (6.4) in the neighbourhood of Γ_ε^*, we obtain the relation

$$p(\varepsilon; \xi_1, \xi_2) \simeq -(2\pi)^{-1/2} k^{0*}(\sigma)\sqrt{\nu - h_\varepsilon^*(\sigma)}, \quad \nu \geqslant h_\varepsilon^*(\sigma) \tag{7.2}$$

Relations (7.1) and (7.2) are respectively the outer and inner asymptotic expansions. The following formula, which combines (7.1) and (7.2), gives the uniformly valid asymptotic representation for the contact pressure

$$p(\varepsilon; \xi_1, \xi_2) \simeq p_0\sqrt{1 - \frac{\xi_1^2}{(a^*)^2} - \frac{\xi_2^2}{(a^*)^2(1-e^2)} + \varepsilon^5 C^*(\xi_1, \xi_2)} \tag{7.3}$$

On achieving coincidence of the two-term asymptotic form (7.3) with (7.1), we obtain

$$C^*(\xi_1, \xi_2) = -4\pi(a^*)^2(3\bar{Q}^*)^{-1}\sqrt{1-c^2}(c_{00}^* + c_{20}^*\xi_1^2 + c_{02}^*\xi_2^2 + c_{40}^*\xi_1^4 + c_{22}^*\xi_1^2\xi_2^2 + c_{04}^*\xi_2^4) \tag{7.4}$$

In this case, the behaviour of the density (7.3) at the boundary of the contact area is characterized by formula (7.2).

Note that, on transforming the density of the contact pressures (7.3) to the initial coordinates x_1 and x_2, it is necessary to recall formulae (3.1) and $\partial/\partial x_i = \varepsilon^{-1}\partial/\partial\xi_i$, relations (1.7) and the following: $p(x_1, x_2) \simeq -\sigma_{33}(\mathbf{W}; \xi_1, \xi_2, 0)$ (see (1.7) and (3.8)).

It has been shown above that the asymptotic form of the contact force $Q \simeq \varepsilon^2 Q^* + \varepsilon^7 q^*$ is found using outer asymptotic expansion (7.1). Avoiding the calculation of the density (5.2), an explicit expression for the correction q^* can be obtained in terms of the right-hand side of integral equation (5.1) using Mossakovskii's theorem (see [28] and, also, [26, § 10, Chapter 2]) in the form

$$q^* = -\frac{\mu}{1-\nu}\left(a^*\sqrt{1-e^2}\mathbf{K}(e)\right)^{-1}\iint\limits_{\omega_0^*} \frac{\varphi_0^*(\xi_1, \xi_2)}{H^*(\xi_1, \xi_2)} d\xi_1 d\xi_2 \tag{7.5}$$

Using the reciprocity theorem, it was shown for the first time in [28] that it is possible, in addition, to obtain a formula similar to (7.5) for the second-order polymoments

$$m_{2,n}^* = -\frac{\mu a^*}{2(1-\nu)\sqrt{1-e^2}}\iint\limits_{\omega_0^*} \frac{\Pi_{2,n}^*(\xi_1, \xi_2)\varphi_0^*(\xi_1, \xi_2)}{H^*(\xi_1, \xi_2)} d\xi_1 d\xi_2 \tag{7.6}$$

From known results [3, 29], we find

$$\Pi_{2,0}^*(\xi_1, \xi_2) = \frac{1}{3\mathbf{K}(e)} - \sum_{i=1}^{2} \pi_i^*(\xi_1, \xi_2)$$

$$\Pi_{0,2}^*(\xi_1, \xi_2) = \frac{1-e^2}{3\mathbf{K}(e)} - \sum_{i=1}^{2} (3\sigma_i - 1)\pi_i^*(\xi_1, \xi_2)$$

$$\pi_i^*(\xi_1,\xi_2)=C_i(e)[\sigma_i+(a^*)^{-2}\xi_1^2-(3\sigma_{3-i}-1)^{-1}(a^*)^{-2}\xi_1^2]$$

$$C_1(e)=\frac{2(\sigma_1-e^2)^2}{(3\sigma_1^2-e^2)(1-\sigma_1)[\mathbf{K}(e)-3\sigma_2\mathbf{D}(e_1)]}$$

$$C_2(e)=\frac{2\sigma_1^2(3\sigma_1-1)^2}{(3\sigma_1-e_1)(3\sigma_1^2-e^2)[\sigma_1\mathbf{C}(e)+(2\sigma_1-1)\mathbf{D}(e)]}$$

$$3\sigma_{1,2}=1+e^2\pm\sqrt{1-e^2+e^4};\quad \mathbf{C}(e)=e^{-2}[2\mathbf{D}(e)-\mathbf{K}(e)]$$

Integrals (7.5) and (7.6) are easily evaluated using formula (52.21) in [3]. We also note that the simple result

$$\Pi_{2,1}^*(\xi_1,\xi_2)=2[\mathbf{D}(e)-\mathbf{C}(e)]^{-1}(a^*)^{-2}\xi_1\xi_2$$

is obtained for the mixed polymoments $m_{2,1}^*$.

Remark. Suppose a punch is indented into an elastic layer which is bounded by the surface

$$x_3=(2R_1)^{-1}x_1^2+(2R_2)^{-1}x_2^2+b_{40}x_1^4+b_{22}x_1^2x_2^2+b_{04}x_2^4$$

Then, from the preceding considerations, we have the following approximate formula, which refines (1.8), for the contact pressure under the punch

$$p(x_1,x_2)=\frac{3Q}{2\pi a^2\sqrt{1-e^2}}\sqrt{1-\frac{x_1^2}{a^2}-\frac{x_2^2}{a^2(1-e^2)}}+C(x_1,x_2) \tag{7.7}$$

where the set Q, a and e is the solution of problem (1.9), (1.10). The fourth-degree polynomial $C(x_1, x_2)$ in (7.7) is recovered from formula (7.4) using solution (5.2) of integral equation (5.1) with a right-hand side $-[\varphi_0^*(\xi_1,\xi_2)+b_{40}^*\xi_1^4+b_{22}^*\xi_1^2\xi_2^2+b_{04}^*\xi_2^4]$. At the same time, the asterisks are disposed of and the variables ξ_1, ξ_2 are replaced by x_1, x_2 in the last expression and in formulae (7.4), (5.2) and (5.1).

8. THE AXISYMMETRIC CASE

Since, in this case, a solution of the model problem for inner asymptotic expansion (3.8) can be easily constructed using well-known results [19, 29], the expression for the unilateral contact boundary condition (3.9) cannot be simplified as was done in (3.13). Moreover, separating the terms $O(\varepsilon^5)$ in (3.13) only makes the solution more complicated. Therefore, instead of formulae (3.13)–(3.16), in accordance with relations (3.6), (3.9) and (2.4)–(2.7), we shall use the following

$$\delta_0^*-\Phi^*(\xi_1,\xi_2)-\varepsilon V_3^*(\varepsilon;\xi_1,\xi_2,0)=\delta_\varepsilon^*-\Lambda_1^*\rho^2-\Lambda_2^*\rho^4$$

$$\delta_\varepsilon^*=\delta_0^*+\varepsilon\frac{a_0}{H}\tilde{Q}^*+\varepsilon^3\frac{2a_1}{H^3}\tilde{\mathcal{M}}_2^*+\varepsilon^5\frac{24a_2}{H^5}\tilde{\mathcal{M}}_4^* \tag{8.1}$$

$$\Lambda_1^*=\frac{1}{2R^*}-\varepsilon^3\frac{a_1}{H^3}\tilde{Q}^*-\varepsilon^5\frac{8a_2}{H^5}\tilde{\mathcal{M}}_2^*,\quad \Lambda_2^*=-\varepsilon^5\frac{a_2}{H^5}\tilde{Q}^* \tag{8.2}$$

Here, $\rho=(\xi_1^2+\xi_2^2)^{1/2}$ is the polar radius in the system of coordinates (3.1) and $\mathcal{M}_m^*$ is the polar polymoment or order m, calculated using the formula (compare with (3.11))

$$\mathcal{M}_m^*=-\frac{1}{m!}\iint_{\omega_\varepsilon^*}(\xi_1^2+\xi_2^2)^{m/2}\sigma_{33}(\mathbf{W};\xi_1,\xi_2,0)d\xi_1d\xi_2,\quad m=2,4 \tag{8.3}$$

Using the formulae in [19, Chapter 3, § 2], we find the following expression for the density of the contact pressures

$$p(\varepsilon;\rho)=\frac{8\mu a^*}{\pi(1-\nu)}\sqrt{1-\frac{\rho^2}{(a^*)^2}}\left(\Lambda_1^*+\frac{8}{9}(a^*)^2\Lambda_2^*+\frac{16}{9}\Lambda_2^*\rho^2\right) \tag{8.4}$$

Here a^* is the radius of the contact area, to determine which we have the equation

$$\delta_\varepsilon^* = 2\Lambda_1^*(a^*)^2 + \frac{8}{3}\Lambda_2^*(a^*)^4 \tag{8.5}$$

On integrating the density (8.4), we fix the first matching condition (3.1)

$$\mathcal{Q}^* = \frac{16\mu}{3(1-\nu)}(a^*)^3\left(\Lambda_1^* + \frac{8}{5}(a^*)^2\Lambda_2^*\right) \tag{8.6}$$

In the axisymmetric case, matching conditions (3.1) are replaced by (8.3). Substituting expression (8.4) into (8.3), we obtain

$$\mathcal{M}_2^* = \frac{16\mu(a^*)^5}{15(1-\nu)}\left(\Lambda_1^* + \frac{40}{21}(a^*)^2\Lambda_2^*\right), \quad \mathcal{M}_4^* = \frac{16\mu(a^*)^7}{315(1-\nu)}\left(\Lambda_1^* + \frac{56}{27}(a^*)^2\Lambda_2^*\right) \tag{8.7}$$

The system of equations (8.5)–(8.7) serves to determine the quantities $a^*, d, \mathcal{M}_2^*$ and $\mathcal{M}_4^*$. We substitute the second expression of (8.1) and expression (8.2) into relations (8.5)–(8.7) and, neglecting quantities of a higher order of smallness compared with those retained in (8.1) and (8.2), we calculate the equations relating the quantities δ_0^* and a^*, $\mathcal{Q}^*$. Thus, by confining ourselves to the leading terms of the asymptotic form in (8.7), we transform Eqs (8.5) and (8.6) to the form

$$\delta_0^* = \frac{(a^*)^2}{R^*} - \varepsilon\frac{a_0}{H}\tilde{\mathcal{Q}}^* - \varepsilon^3\frac{16a_1}{5\pi}\frac{(a^*)^5}{H^3R^*} - \varepsilon^5\frac{512a_2}{63\pi}\frac{(a^*)^7}{H^5R^*} \tag{8.8}$$

$$\tilde{\mathcal{Q}}^* = \frac{4}{3\pi}\frac{(a^*)^3}{R^*}\left(1 - \varepsilon^3\frac{8a_1}{3\pi}\frac{(a^*)^3}{H^3} - \varepsilon^5\frac{128a_2}{15\pi}\frac{(a^*)^5}{H^5}\right) \tag{8.9}$$

The following will be the consequence of Eqs (8.8) and (8.9)

$$\tilde{\mathcal{Q}}^* = \frac{4}{3\pi}\delta_0^*a^*\left(1 - \varepsilon\frac{4a_0}{3\pi}\frac{a^*}{H}\right)^{-1}\left(1 + \frac{(a^*)^2}{\delta_0^*R^*}\left[\varepsilon^3\frac{8a_1}{15\pi}\frac{(a^*)^3}{H^3} - \varepsilon^5\frac{128a_2}{315\pi}\frac{(a^*)^5}{H^5}\right]\right) \tag{8.10}$$

Following the approach described earlier [3], we now introduce the large parameter $\lambda = H/a$, where $a = \varepsilon a^*$ is the actual radius of the contact area. Then, taking account of relations (1.2) and (2.8), we write Eq. (8.9) in the form

$$\tilde{\mathcal{Q}} = \frac{4}{3\pi}\frac{a^3}{R}\left(1 - \lambda^{-3}\frac{8a_1}{3\pi} - \lambda^{-5}\frac{128a_2}{15\pi}\right) \tag{8.11}$$

Finally, using equalities (8.8) and (8.9), Eq. (8.10) takes the form

$$\tilde{\mathcal{Q}} = \frac{4}{3\pi}\delta_0 a\left[\left(1 - \lambda^{-1}\frac{4a_0}{3\pi}\right)^{-1} + \lambda^{-3}\frac{8a_1}{15\pi}\left(1 + \lambda^{-1}\frac{8a_0}{3\pi}\right) + \lambda^{-5}\frac{128}{45\pi}\left(\frac{a_0a_1}{\pi} - \frac{a_2}{7}\right)\right] \tag{8.12}$$

Relations (8.11) and (8.12) refine the well-known result in [3] (see formulae (49.2 and 49.3)).

Remark. The unusual "concept of the structural stability of the solution of a contact problem in the linear theory of elasticity as the conservation of diffeomorphic equivalence by the contact pressure in the case of a small perturbation in the shapes of bodies" was introduced in [30]. The example, constructed in [30] of structural stability in an axisymmetric problem for a punch with a form function $\Phi(\rho) = \Lambda_1\rho^2 + \Lambda_2\rho^4$ is devoid of mechanical content.

Actually, it is well known [19] (also, see [31]) that the equation, relating the displacement of a punch δ_0 with radius of the contact area a, has the form of (8.5). The positive root of the quadratic equation (8.5) can be transformed to the form

$$a^2 = \delta_0(\Lambda_1 + \sqrt{\Lambda_1^2 + {}^8\!/_3\Lambda_2\delta_0})^{-1} \tag{8.13}$$

As in [30], we assume that the initial shape of the punch is specified by the term $\Lambda_1\rho^2$ and that the term $\Lambda_2\rho^4$ will be the perturbed shape. Then, for a fixed value for the embedding depth of the punch δ_0, the value of the right-hand side of equality (8.13) will differ only slightly from the specified value $\delta_0/(2\Lambda_1)$ for any sufficiently small perturbation. We also arrive at this conclusion in the case of a fixed value of the force Q which impresses the punch (see Eq. (8.6)).

However, if, as was done in [30], the constraint

$$8\Lambda_2 a^2 = 3\Lambda_1 \tag{8.14}$$

is initially imposed on the quantities a, Λ_1 and Λ_2, then $a^2 = 3\Lambda_1/(8\Lambda_2)$ follows from the same equation (8.14). We therefore obtain, as a consequence of the last equality, that the value a of the radius of the contact area, and "together with it the compressive stress can be very large in the case of a small perturbation", that is, when the magnitude of Λ_2 is very small [30]. We emphasize that this inference is obtained directly from constraint (8.14), imposed *a priori* on the magnitudes of a, Λ_1 and Λ_2, which arose in [30] from the requirement (which does not make any mechanical sense) that the second derivative of the density of the contact pressure (8.4) with respect to the radius should vanish at the centre of the contact area.

9. CONCLUSION

We recall that the approximate solution of the contact problem for an elastic layer which has been constructed also remains valid, for example, in the case of the pressure of a punch on the centre of a circular elastic plate. It is only necessary to calculate the values of the corresponding coefficients (see (2.5)–(2.7)), the characteristic dimensions of the plate and the conditions under which it is clamped.

It is clear that the effectiveness of the implementation of the method used [21] when solving the model problem of unilateral contact rests on the possibility of constructing the solution of a contact problem with an elliptic contact area in explicit form. The question of the further refinement of the results which have been obtained therefore remains open. In the case of the axisymmetric problem, no fundamental difficulties are encountered in constructing the next approximations.

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